Inequalities - Winter Camp 2009

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1. Theorems that Are good to know

A lot of mileage can be gotten from the basic fact that the square of a real number a is non-negative. For instance,
$(a+b)^2 \geq 0 \rightarrow a^2 + b^2 \geq 2ab$
and we immediately derive the basic **Am-Gm** inequality. Likewise, consider the famous

$$\mathbf{Cauchy - Schwartz - Inequality} : (\sum_{\mathbf{i}}^{\mathbf{n}} \mathbf{a_i^2})(\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{b_i^2}) \geq (\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{a_i b_i})^{\mathbf{2}}$$

To prove the inequality, just note that we can rewrite the RHS-LHS as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2$$

which, being a sum of squares, is clearly positive. Another way to generalize the Am-Gm inequality is the following

Am-Gm revised:

$$x_i \geq 0 \rightarrow \frac{x_1 + x_2 + ... + x_n}{n} \geq (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$$

This generalized Am-Gm inequality can be deduced from the classical one easily for n a power of 2. For example, applying the classical Am-Gm twice, we get:

$a^4 + b^4 + c^4 + d^4 \geq 2(a^2b^2 + c^2d^2) \geq 4abcd$ which yields the case of $n = 4$. For powers of 2 one can repeat the same trick. For non-powers of 2, the best way to understand the above is by a special case of the equally useful

Jensen's inequality *Suppose f(x) is a continuos function satisfying* $\frac{f(x)+f(y)}{2} \geq f(\frac{x+y}{2})$ *for all x,y. Then for all* x_i, $\sum_{i=1}^{n} f(x_i) \geq nf(\frac{x_1 + x_2 + ... + x_n}{n})$.
The condition $f(x) + f(y) \geq 2f(\frac{x+y}{2})$ *is known as* convexity*, and we say that f is a* convex *function.*

To become proficient at Jensen's, it's good to know a bunch of convex functions. Here are 2 very useful examples. Note that if $f(x)$ is convex, than $cf(x)$ is convex as long as $c \geq 0$. Negative c aren't allowed!!!

1. x^n for $n \geq 1$ or $n \leq 0$ and $x \geq 0$
2. $-sin(x)$ for $0 \leq x \leq \pi$

Here are some more useful generalizations of Am-Gm.

Weighted Am-Gm for $w_i \geq 0$ and $x_i \geq 0, \sum_{i=1}^{n} w_i x_i \geq \prod_{i=1}^{n} x_i^{w_i}$

Remark

The weighted Am-Gm inequality is in some sense the most natural and convenient phrasing of Am-Gm. Try proving it (or at least getting the idea of the proof, which is at leas as important) by first doing the case of rational weights w_i, and then deducing it for reals by continuity.

Holders inequality for $a_i, b_i, \alpha, \beta \geq 0, (\sum_{i=1}^{n} a_i)^{\alpha} (\sum_{i=1}^{n} b_i)^{\beta} \geq (\sum_{i=1}^{n} a_i^{\frac{\alpha}{\alpha+\beta}} b_i^{\frac{\beta}{\alpha+\beta}})^{\alpha+\beta}$

It is easy to get discouraged by the the generality of Holder's inequality, but in practise it very clean and intuitive, so it's good to play around with examples so as to get comfortable with it. For example, use Holders to prove the following

Excercise: Prove $(a_1 + a_2)(b_1 + b_2)(c_1 + c_2) \geq ((a_1 b_1 c_1)^{\frac{1}{3}} + (a_2 b_2 c_2)^{\frac{1}{3}})^3$, if all variables are positive.

Power-Mean inequality If $x \geq y > 0$,and $a_1, a_1, ., a_n \geq 0$, then

$(\frac{a_1^x + a_2^x + ... + a_n^x}{n})^{\frac{1}{x}} \geq (\frac{a_1^y + a_2^y + ... + a_n^y}{n})^{\frac{1}{y}}$

This is actually a generalization of AM-GM, in the limiting case of $x = 1, y = 0$

No inequality toolbox is complete without knowing Schur's inequality, which is a nice bridge from the Jensen-type inequalities we've been studying to rearrangement inequalities.

Schur's Inequality If $a, b, c \geq 0$, then

$a^3 + b^3 + c^3 + 3abc \geq a^2 b + b^2 a + a^2 c + c^2 a + b^2 c + c^2 b$

An indication that Schur's is powerful is that it cannot be deduced from Am-Gm, due to the 'weak' term abc on the LHS. It is very instructive to go through the proof of Schur's. To start, break symmetry by saying WLOG, $a \geq b \geq c$. The equation then rearranges as

$a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) \geq 0$

which we can then rearrange to

$(a - b)(a(a - c) - b(b - c)) + c(c - a)(c - b) \geq 0$

but this is clearly true, since every term on the LHS is positive(convince yourself of this!) The proof is very short, but these sort of proofs are tricky to come up with, since by ordering the variables the symmetry is lost. There are a few general inequalities based on the idea of ordering the variables.

Rearrangement inequality if $a_1 \leq a_2 \leq ... \leq a_n$ and $b_1 \leq b_2 \leq ... \leq b_n$ and π is any permutation of $1, 2, ..., n$ then $\displaystyle\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\pi(i)}$

Chebychev's inequality With same assumptions as above, then $\displaystyle\sum_{i=1}^{n} a_i b_i \geq \frac{(a_1 + a_2 + ... + a_n)(b_1 + b_2 .. + b_n)}{n}$

Chebychev can of course be deduced with n applications of the rearrangement inequality. One of the reasons that the above are so powerful is they make **no assumptions on positivity!**

PROBLEMS

1. Prove for $x \geq 0$, we have $x^2 - 3 \geq -\frac{2}{x}$
2. Prove that for $p, q, r, a, b, c \geq 0, p + q + r = 1$,

   $a + b + c \geq \sum_{cyc} a^p b^q c^r$
3. Prove that for $a, b, c, d \geq 0, abcd = 1$,

   $a^2 + b^2 + c^2 + d^2 \geq 4$
4. Prove that for $a_i \geq 0$,
   $\displaystyle\prod_{i=1}^{n}(1 + a_i) \geq (1 + (a_1 a_2 .. a_n)^{\frac{1}{n}})^n$.
5. (APMO 2004/5) Prove that if $x, y, z \geq 0$ then $(2 + x^2)(2 + y^2)(2 + z^2) \geq 9(xy + xz + xz)$
6. Let $a_1, a_2, .., a_n$ be positive reals with sum 1. Prove that

   $a_1 a_2 + a_2 a_3 + ... + a_{n-1} a_n \leq \frac{1}{4}$ and determine the cases of equality.
7. Prove that if $a, b, c \geq 0$ with $abc = 1$ then

   $\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ac}{a^5+c^5+ac} \leq 1$
8. Let $a, b, c \geq 0, a + b + c = 1$ Prove that

   $\sqrt{ab + c} + \sqrt{ac + b} + \sqrt{bc + a} \geq 1 + \sqrt{ac} + \sqrt{bc} + \sqrt{ac}$
9. Let $a, b, c \geq 0$ Prove that

   $\frac{1}{a^3+b^3+abc} + \frac{1}{a^3+c^3+abc} + \frac{1}{b^3+c^3+abc} \leq \frac{1}{abc}$
10. (IMO 1975) If a, b, c, d are positive reals, prove that $1 < \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} \leq 2$, and all intermediate values are achieved.

11. (RUSSIA 2004) $x_1, x_2, ..., x_n, n > 3$ are positive real numbers with product 1. Prove that
$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + ... + \frac{1}{1+x_n+x_nx_1} \geq 1$

    HINT: the product 1 condition is annoying. Can you reparametrize and get rid of it?

12. (Japan 2005/3) Let $a, b, c \geq 0, a+b+c = 1$. Prove that

    $a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b} \leq 1$

13. Supppose the real quadratic form $Q(\mathbf{x}) = \sum_{i,j=1}^{n} a_{i,j}x_ix_j$ is non-negative for all real numbers x_i, x_j. Prove you can write $Q(\mathbf{x})$ as a sum of squares of linear forms.

14. (Baltic way 04) Prove that if p, q, r are positive real numbers with product 1,

    then for all natural numbers n we have

    $\frac{1}{p^n+q^n+1} + \frac{1}{p^n+r^n+1} + \frac{1}{q^n+r^n+1} \leq 1$

15. (Iran 1996)Prove the following inequality for positive real numbers x,y,z:

    $(xy + xz + yz)(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(x+z)^2}) \geq \frac{9}{4}$

16. (Russia ??) $a, b, c, d \geq 0$, prove that

    $(ab)^{\frac{1}{3}} + (cd)^{\frac{1}{3}} \leq (a+c+d)^{\frac{1}{3}}(a+c+b)^{\frac{1}{3}}$

17. (Mathlinks) $x, y, z \geq 0$, prove

    $\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \leq \frac{3}{2}\frac{x^2+y^2+z^2}{xy+xz+yz}$

18. (Baltic way 08) a, b, c are positive real numbers with $a^2 + b^2 + c^2 = 3$. Prove that
$\frac{a^2}{2+b+c^2} + \frac{b^2}{2+c+a^2} + \frac{c^2}{2+a+b^2} \geq \frac{(a+b+c)^2}{12}$

19. (Usamo 00/6) DISCLAIMER: THE FOLLOWING PROBLEM IS CONSIDERED TO BE EXTREMELY DIFFICULT!

    If $0 \leq a_1 \leq a_2 ... \leq a_n$ and $0 \leq b_1 \leq b_2 ... \leq b_n$, then show that

    $\sum_{i,j} MIN(a_ia_j, b_ib_j) \leq \sum_{i,j} MIN(a_ib_j, b_ia_j)$

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